On Brauer p-dimensions and index-exponent relations over finitely-generated field extensions*

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Abstract

Let E be a field of absolute Brauer dimension $\mathrm{abrd}(E)$, and F/E a transcendental finitely-generated extension. This paper shows that the Brauer dimension $\mathrm{Brd}(F)$ is infinite, if $\mathrm{abrd}(E) = \infty$. When the absolute Brauer p-dimension $\mathrm{abrd}_p(E)$ is infinite, for some prime number p, it proves that for each pair (n, m) of integers with $n \geq m > 0$, there is a central division F-algebra of Schur index p^n and exponent p^m. Lower bounds on the Brauer p-dimension $\mathrm{Brd}_p(F)$ are obtained in some important special cases where $\mathrm{abrd}_p(E) < \infty$. These results solve negatively a problem posed by Auel et al. (Transf. Groups **16**: 219-264, 2011).

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1 Introduction

Let E be a field, $s(E)$ the class of finite-dimensional associative central simple E-algebras, $d(E)$ the subclass of division algebras $D \in s(E)$, and for each $A \in s(E)$, let $[A]$ be the equivalence class of A in the Brauer group $\mathrm{Br}(E)$. It is known that $\mathrm{Br}(E)$ is an abelian torsion group (cf. [34], Sect. 14.4), whence it decomposes into the direct sum of its p-components $\mathrm{Br}(E)_p$, where p runs across the set $\mathbb{P}$ of prime numbers. By Wedderburn's structure theorem (see, e.g., [34], Sect. 3.5), each $A \in s(E)$ is isomorphic to the full matrix ring $M_n(D_A)$ of order n over some $D_A \in d(E)$ that is uniquely determined by A, up-to an E-isomorphism. This implies the dimension $[A\colon E]$ is a square of a positive integer $\deg(A)$, the degree of A. The main numerical invariants of A are $\deg(A)$, the Schur index $\mathrm{ind}(A) = \deg(D_A)$, and the exponent $\exp(A)$, i.e. the order

*Throughout this paper, we write for brevity "FG-extension(s)" instead of "finitely-generated [field] extension(s)".

of $[A]$ in $\mathrm{Br}(E)$. The following statements describe basic divisibility relations between $\mathrm{ind}(A)$ and $\exp(A)$, and give an idea of their behaviour under the scalar extension map $\mathrm{Br}(E) \to \mathrm{Br}(R)$, in case R/E is a field extension of finite degree $[R\colon E]$ (see, e.g., [34], Sects. 13.4, 14.4 and 15.2, and [5], Lemma 3.5):

(1.1) (a) $(\mathrm{ind}(A), \exp(A))$ is a Brauer pair, i.e. $\exp(A)$ divides $\mathrm{ind}(A)$ and is divisible by every $p \in \mathbb{P}$ dividing $\mathrm{ind}(A)$.

(b) $\mathrm{ind}(A \otimes_E B)$ is divisible by $\mathrm{l.c.m.}\{\mathrm{ind}(A), \mathrm{ind}(B)\}/\mathrm{g.c.d.}\{\mathrm{ind}(A), \mathrm{ind}(B)\}$ and divides $\mathrm{ind}(A)\mathrm{ind}(B)$, for each $B \in s(E)$; in particular, if $A, B \in d(E)$ and $\mathrm{g.c.d.}\{\mathrm{ind}(A), \mathrm{ind}(B)\} = 1$, then the tensor product $A \otimes_E B$ lies in $d(E)$.

(c) $\mathrm{ind}(A)$, $\mathrm{ind}(A \otimes_E R)$, $\exp(A)$ and $\exp(A \otimes_E R)$ divide $\mathrm{ind}(A \otimes_E R)[R\colon E]$, $\mathrm{ind}(A)$, $\exp(A \otimes_E R)[R\colon E]$ and $\exp(A)$, respectively.

Statements (1.1) (a), (b) imply Brauer's Primary Tensor Product Decomposition Theorem, for any $\Delta \in d(E)$ (cf. [34], Sect. 14.4), and (1.1) (a) fully describes general restrictions on index-exponent relations, in the following sense:

(1.2) Given a Brauer pair $(m', m) \in \mathbb{N}^2$, there is a field F with $(\mathrm{ind}(D), \exp(D)) = (m', m)$, for some $D \in d(F)$ (Brauer, see [34], Sect. 19.6). One may take as F any rational (i.e. purely transcendental) extension in infinitely many variables over any fixed field F_0 (see also Corollary 4.4 and Remark 4.5).

As in [2], Sect. 4, we say that a field E is of finite Brauer p-dimension $\mathrm{Brd}_p(E) = n$, for a fixed $p \in \mathbb{P}$, if n is the least integer ≥ 0, for which $\mathrm{ind}(D) \leq \exp(D)^n$ whenever $D \in d(E)$ and $[D] \in \mathrm{Br}(E)_p$. If no such n exists, we set $\mathrm{Brd}_p(E) = \infty$. The absolute Brauer p-dimension of E is defined as the supremum $\mathrm{abrd}_p(E) = \sup\{\mathrm{Brd}_p(R)\colon\ R \in \mathrm{Fe}(E)\}$, where $\mathrm{Fe}(E)$ is the set of finite extensions of E in a separable closure E_{sep}. Clearly, $\mathrm{Brd}_p(E) \leq \mathrm{abrd}_p(E)$, $p \in \mathbb{P}$. We say that E is a virtually perfect field, if $\mathrm{char}(E) = 0$ or $\mathrm{char}(E) = q > 0$ and E is a finite extension of its subfield $E^q = \{e^q\colon\ e \in E\}$.

It is known that $\mathrm{Brd}_p(E) = \mathrm{abrd}_p(E) = 1$, for all $p \in \mathbb{P}$, if E is a global or local field (cf. [35], (31.4) and (32.19)), or the function field of an algebraic surface defined over an algebraically closed field E_0 [19], [24] (see also Remark 5.8). As shown in [27], $\mathrm{abrd}_p(E) < p^{n-1}$, $p \in \mathbb{P}$, provided that E is the function field of an n-dimensional algebraic variety defined over an algebraically closed field E_0. Similarly, $\mathrm{abrd}_p(E) < p^n$, $p \in \mathbb{P}$, if E_0 is a finite field, the maximal unramified extension of a local field, or a perfect pseudo algebraically closed (PAC) field (for the C_1-type of E_0, used in [27] for proving these inequalities, see [22] and [21], [15], Theorem 21.3.6, respectively). The suprema $\mathrm{Brd}(E) = \sup\{\mathrm{Brd}_p(E)\colon\ p \in \mathbb{P}\}$ and $\mathrm{abrd}(E) = \sup\{\mathrm{Brd}(R)\colon\ R \in \mathrm{Fe}(E)\}$ are called a Brauer dimension and an absolute Brauer dimension of E, respectively. In view of (1.1), the definition of $\mathrm{Brd}(E)$ is the same as the one given in [2], Sect. 4. It has recently been proved [16], [33] (see also [8], Propositions 6.1 and 7.1), that $\mathrm{abrd}(K_m) < \infty$, provided $m \in \mathbb{N}$ and (K_m, v_m) is an m-dimensional local field, in the sense of [14], with a finite m-th residue field $\widehat{K}_m$.

The present research is devoted to the study of index-exponent relations over transcendental FG-extensions F of a field E and their dependence on $\mathrm{abrd}_p(E)$, $p \in \mathbb{P}$. It is motivated mainly by two questions concerning the dependence of $\mathrm{Brd}(F)$ upon $\mathrm{Brd}(E)$, stated as open problems in Sect. 4 of the survey [2].

2 The main results

While the study of index-exponent relations makes interest in its own right, it should be noted that fields E with $\mathrm{abrd}_p(E) < \infty$, for all $p \in \mathbb{P}$, are singled out by Galois cohomology (see [20] and [40], as well as [27], Sects. 5-8, and further references in [7], Remark 4.2). It is also worth mentioning the following fact about the almost perfect fields of this type (see [4], [5], and Lemma 4.1):

(2.1) Every locally finite dimensional associative central division E-algebra R possesses an E-subalgebra $\widetilde{R}$ with the following properties:

(a) $\widetilde{R}$ decomposes into a tensor product $\otimes_{p\in\mathbb{P}} R_p$, where $\otimes = \otimes_E$, $R_p \in d(E)$ and $[R_p] \in \mathrm{Br}(E)_p$, for each $p \in \mathbb{P}$;

(b) Finite-dimensional E-subalgebras of R are embeddable in $\widetilde{R}$;

(c) $\widetilde{R}$ is isomorphic to R, if the dimension $[R\colon E]$ is countably infinite.

It would be of definite interest to know whether function fields of algebraic varieties over a global, local or algebraically closed field are of finite absolute Brauer dimensions. This draws our attention to the following open question:

(2.2) Is the class of fields E of finite absolute Brauer p-dimensions, for a fixed $p \in \mathbb{P}$, $p \neq \mathrm{char}(E)$, closed under the formation of FG-extensions?

The main result of this paper shows, for a transcendental FG-extension F/E, the strong influence of p-dimensions $\mathrm{abrd}_p(E)$ on $\mathrm{Brd}_p(F)$, and on index-exponent relations over F, as follows:

Theorem 2.1. *Let E be a field, $p \in \mathbb{P}$ and F/E an FG-extension of transcendency degree $\mathrm{trd}(F/E) = \kappa \geq 1$. Then:*

(a) $\mathrm{Brd}_p(F) \geq \mathrm{abrd}_p(E) + \kappa - 1$, *if* $\mathrm{abrd}_p(E) < \infty$ *and F/E is rational;*

(b) *If* $\mathrm{abrd}_p(E) = \infty$, *then* $\mathrm{Brd}_p(F) = \infty$ *and for each $n, m \in \mathbb{N}$ with $n \geq m > 0$, there exists $D_{n,m} \in d(F)$ with* $\mathrm{ind}(D_{n,m}) = p^n$ *and* $\exp(D_{n,m}) = p^m$*;*

(c) *$Brd_p(F) = \infty$, provided $p = \mathrm{char}(E)$ and $[E\colon E^p] = \infty$; if $\mathrm{char}(E) = p$ and $[E\colon E^p] = p^\nu < \infty$, then $\nu + \kappa - 1 \leq \mathrm{Brd}_p(F) \leq \mathrm{abrd}_p(F) \leq \nu + \kappa$.*

It is known (cf. [23], Ch. X) that each FG-extension F of a field E possesses a subfield F_0 that is rational over E with $\mathrm{trd}(F_0/E) = \mathrm{trd}(F/E)$. This ensures that $[F\colon F_0] < \infty$, so (1.1) and Theorem 2.1 imply the following:

(2.3) If (2.2) has an affirmative answer, for some $p \in \mathbb{P}$, $p \neq \mathrm{char}(E)$, and each FG-extension F/E with $\mathrm{trd}(F/E) = \kappa \geq 1$, then there exists $c_\kappa(p) \in \mathbb{N}$, depending on E, such that $\mathrm{Brd}_p(\Phi) \leq c_\kappa(p)$, for every FG-extension Φ/E with $\mathrm{trd}(\Phi/E) < \kappa$. For example, this applies to $c_k(p) = \mathrm{Brd}_p(E_\kappa)$, where E_κ/E is a rational FG-extension with $\mathrm{trd}(E_\kappa/E) = \kappa$.

The application of Theorem 2.1 is facilitated by the following result of [7] (see Example 6.2 below, for an alternative proof in characteristic zero):

Proposition 2.2. *For each $q \in \mathbb{P} \cup \{0\}$ and $k \in \mathbb{N}$, there exists a field $E_{q,k}$ with $char(E_{q,k}) = q$, $Brd(E_{q,k}) = k$ and $abrd_p(E_{q,k}) = \infty$, for all $p \in \mathbb{P} \setminus P_q$, where $P_0 = \{2\}$ and $P_q = \{p \in \mathbb{P}\colon\ p \mid q(q-1)\}$, $q \in \mathbb{P}$. Moreover, if $q > 0$, then $E_{q,k}$ can be chosen so that $[E_{q,k}\colon E_{q,k}^q] = \infty$.*

Theorem 2.1, Proposition 2.2 and statement (1.1) (b) imply the following:

(2.4) There exist fields E_k, $k \in \mathbb{N}$, such that $\mathrm{char}(E_k) = 2$, $\mathrm{Brd}(E_k) = k$ and all Brauer pairs $(m', n') \in \mathbb{N}^2$ are index-exponent pairs over any transcendental FG-extension of E_k.

It is not known whether (2.4) holds in any characteristic $q \neq 2$. This is closely related to the following open problem:

(2.5) Find whether there exists a field E containing a primitive p-th root of unity, for a given $p \in \mathbb{P}$, such that $\mathrm{Brd}_p(E) < \mathrm{abrd}_p(E) = \infty$.

Statement (1.1) (b), Theorem 2.1 and Proposition 2.2 imply the validity of (2.4) in zero characteristic, for Brauer pairs of odd positive integers. When $q > 2$, they show that if $[E_{q,k} \colon E_{q,k}^q] = \infty$, then Brauer pairs $(m', m) \in \mathbb{N}^2$ relatively prime to $q - 1$ are index-exponent pairs over every transcendental FG-extension of $E_{q,k}$. This solves in the negative [2], Problem 4.4, proving (in the strongest presently known form) that the class of fields of finite Brauer dimensions is not closed under the formation of FG-extensions.

Theorem 2.1 (a) makes it easy to prove that the solution to [2], Problem 4.5, on the existence of a "good" definition of a dimension $\dim(E) < \infty$, for some fields E, is negative whenever $\mathrm{abrd}(E) = \infty$ (see Corollary 5.4). It implies that if Problem 4.5 of [2] is solved affirmatively, for all FG-extensions F/E, then each F satisfies the following stronger inequalities than those conjectured by (2.3) (see also Remark 5.5 and [2], Sect. 4):

(2.6) $\mathrm{Brd}(F) < \dim(F)$, $\mathrm{abrd}(F) \leq \dim(F)$ and $\mathrm{abrd}(F) \leq \mathrm{Brd}(E_{t+1}) \leq \mathrm{abrd}(E) + t + c(E)$, for some integer $c(E) \leq \dim(E) - \mathrm{abrd}(E)$, where $t = \mathrm{trd}(F/E)$, E_{t+1}/E is a rational extension and $\mathrm{trd}(E_{t+1}/E) = t + 1$.

The proof of Theorem 2.1 is based on Merkur'ev's theorem about central division algebras of prime exponent [29], Sect. 4, Theorem 2, and on a characterization of fields of finite absolute Brauer p-dimensions generalizing Albert's theorem [1], Ch. XI, Theorem 3. It strongly relies on results of valuation theory, like theorems of Grunwald-Hasse-Wang type, Morandi's theorem on tensor products of valued division algebras [31], Theorem 1, lifting theorems over Henselian (valued) fields, and Ostrowski's theorem. As shown in [7], Sect. 6, the flexibility of this approach enables one to obtain the following results:

(2.7) (a) There exists a field E_1 with $\mathrm{abrd}(E_1) = \infty$, $\mathrm{abrd}_p(E_1) < \infty$, $p \in \mathbb{P}$, and $\mathrm{Brd}(L_1) < \infty$, for every finite extension L_1/E_1;

(b) For any integer $n \geq 2$, there is a Galois extension L_n/E_n, such that $[L_n \colon E_n] = n$, $\mathrm{Brd}_p(L_n) = \infty$, for all $p \in \mathbb{P}$, $p \equiv 1 (\mathrm{mod}\ n)$, and $\mathrm{Brd}(M_n) < \infty$, provided that M_n is an extension of E in $L_{n,\mathrm{sep}}$ not including L_n.

Our basic notation and terminology are standard. For any field K with a Krull valuation v, unless stated otherwise, we denote by $O_v(K)$, $\widehat{K}$ and $v(K)$ the valuation ring, the residue field and the value group of (K, v), respectively; $v(K)$ is supposed to be an additively written totally ordered abelian group. As usual, $\mathbb{Z}$ stands for the additive group of integers, $\mathbb{Z}_p$, $p \in \mathbb{P}$, are the additive groups of p-adic integers, and $[r]$ is the integral part of any real number $r \geq 0$. We write $I(\Lambda'/\Lambda)$ for the set of intermediate fields of a field extension Λ'/Λ, and $\mathrm{Br}(\Lambda'/\Lambda)$ for the relative Brauer group of Λ'/Λ. By a Λ-valuation of Λ',

we mean a Krull valuation v with $v(\lambda) = 0$, for all $\lambda \in \Lambda^*$. Given a field E and $p \in \mathbb{P}$, $E(p)$ denotes the maximal p-extension of E in $E_{\rm sep}$, and $r_p(E)$ the rank of the Galois group $\mathcal{G}(E(p)/E)$ as a pro-p-group ($r_p(E) = 0$, if $E(p) = E$). Brauer groups are considered to be additively written, Galois groups are viewed as profinite with respect to the Krull topology, and by a homomorphism of profinite groups, we mean a continuous one. We refer the reader to [13], [18], [23], [34] and [39], for any missing definitions concerning valuation theory, field extensions, simple algebras, Brauer groups and Galois cohomology.

The rest of the paper proceeds as follows: Sect. 3 includes preliminaries used in the sequel. Theorem 2.1 is proved in Sects. 4 and 5. In Sect. 6 we show that the answer to (2.2) will be affirmative, if this is the case in zero characteristic. Lower bounds on $\mathrm{Brd}_p(F)$ are also obtained in these sections, for FG-extensions F of some frequently used fields E with $\mathrm{abrd}_p(E) < \infty$.

3 Preliminaries on valuation theory

The results of this section are known and will often be used without an explicit reference. We begin with a lemma essentially due to Saltman [36].

Lemma 3.1. *Let (K, v) be a height 1 valued field, K_v a Henselization of K in $K_{\rm sep}$ relative to v, and $\Delta_v \in d(K_v)$ an algebra of exponent $p \in \mathbb{P}$. Then there exists $\Delta \in d(K)$ with $\exp(\Delta) = p$ and $[\Delta \otimes_K K_v] = [\Delta_v]$.*

Proof. By [29], Sect. 4, Theorem 2, Δ_v is Brauer equivalent to a tensor product of degree p algebras from $d(K_v)$, so one may consider only the case of $\deg(\Delta_v) = p$. Then, by Saltman's theorem (cf. [36]), there exists $\Delta \in d(K)$, such that $\deg(\Delta) = p$ and $\Delta \otimes_K K_v$ is K_v-isomorphic to Δ_v, which proves Lemma 3.1. □

In what follows, we shall use the fact that the Henselization K_v of a field K with a valuation v of height 1 is separably closed in the completion of K relative to the topology induced by v (cf. [13], Theorem 15.3.5 and Sect. 18.3). For example, our next lemma is a consequence of Galois theory, this fact and Lorenz-Roquette's valuation-theoretic generalization of Grunwald-Wang's theorem (cf. [23], Ch. VIII, Theorem 4, and [26], page 176 and Theorems 1 and 2).

Lemma 3.2. *Let F be a field, $S = \{v_1, \dots, v_s\}$ a finite set of non-equivalent height 1 valuations of F, and for each index j, let F_{v_j} be a Henselization of K in $K_{\rm sep}$ relative to v_j, and L_j/F_{v_j} a cyclic field extension of degree p^{μ_j}, for some $p \in P$ and $\mu_j \in \mathbb{N}$. Put $\mu = \max\{\mu_1, \dots, \mu_s\}$, and in the case of $p = 2$ and* $\mathrm{char}(F) = 0$*, suppose that the extension $F(\delta_\mu)/F$ is cyclic, where $\delta_\mu \in F_{\rm sep}$ is a primitive 2^μ-th root of unity. Then there is a cyclic field extension L/F of degree p^μ, whose Henselization $L_{v'_j}$ is F_{v_j}-isomorphic to L_j, where v'_j is a valuation of L extending v_j, for $j = 1, \dots, s$.*

Assume that $K = K_v$, or equivalently, that (K, v) is a Henselian field, i.e. v is a Krull valuation on K, which extends uniquely, up-to an equivalence, to a

valuation v_L on each algebraic extension L/K. Put $v(L) = v_L(L)$ and denote by $\widehat{L}$ the residue field of (L, v_L). It is known that $\widehat{L}/\widehat{K}$ is an algebraic extension and $v(K)$ is a subgroup of $v(L)$. When $[L\colon K]$ is finite, Ostrowski's theorem states the following (cf. [13], Theorem 17.2.1):

(3.1) $[\widehat{L}\colon \widehat{K}]e(L/K)$ divides $[L\colon K]$ and $[L\colon K][\widehat{L}\colon \widehat{K}]^{-1}e(L/K)^{-1}$ is not divisible by any $p \in \mathbb{P}$ different from $\mathrm{char}(\widehat{K})$, $e(L/K)$ being the index of $v(K)$ in $v(L)$; in particular, if $\mathrm{char}(\widehat{K}) \dagger [L\colon K]$, then $[L\colon K] = [\widehat{L}\colon \widehat{K}]e(L/K)$.

Statement (3.1) and the Henselity of v imply the following:

(3.2) The quotient groups $v(K)/pv(K)$ and $v(L)/pv(L)$ are isomorphic, if $p \in \mathbb{P}$ and L/K is a finite extension. When $\mathrm{char}(\widehat{K}) \dagger [L\colon K]$, the natural embedding of K into L induces canonically an isomorphism $v(K)/pv(K) \cong v(L)/pv(L)$.

A finite extension R/K is said to be defectless, if $[R\colon K] = [\widehat{R}\colon \widehat{K}]e(R/K)$. It is called inertial, if $[R\colon K] = [\widehat{R}\colon \widehat{K}]$ and $\widehat{R}$ is separable over $\widehat{K}$. We say that R/K is totally ramified, if $[R\colon K] = e(R/K)$; R/K is called tamely ramified, if $\widehat{R}/\widehat{K}$ is separable and $\mathrm{char}(\widehat{K}) \dagger e(R/K)$. The Henselity of v ensures that the compositum K_{ur} of inertial extensions of K in K_{sep} has the following properties:

(3.3) (a) $v(K_{\mathrm{ur}}) = v(K)$ and finite extensions of K in K_{ur} are inertial;

(b) K_{ur}/K is a Galois extension, $\widehat{K}_{\mathrm{ur}} \cong \widehat{K}_{\mathrm{sep}}$ over $\widehat{K}$, $\mathcal{G}(K_{\mathrm{ur}}/K) \cong \mathcal{G}_{\widehat{K}}$, and the natural mapping of $I(K_{\mathrm{ur}}/K)$ into $I(\widehat{K}_{\mathrm{sep}}/\widehat{K})$ is bijective.

Recall that the compositum K_{tr} of tamely ramified extensions of K in K_{sep} is a Galois extension of K with $v(K_{\mathrm{tr}}) = pv(K_{\mathrm{tr}})$, for every $p \in \mathbb{P}$ not equal to $\mathrm{char}(\widehat{K})$. It is therefore clear from (3.1) that if $K_{\mathrm{tr}} \neq K_{\mathrm{sep}}$, then $\mathrm{char}(\widehat{K}) = q \neq 0$ and $\mathcal{G}_{K_{\mathrm{tr}}}$ is a pro-q-group. When this holds, it follows from (3.3) and Galois cohomology (cf. [39], Ch. II, 2.2) that $\mathrm{cd}_q(\mathcal{G}(K_{\mathrm{tr}}/K)) \leq 1$. Hence, by [39], Ch. I, Proposition 16, there is a closed subgroup $\mathcal{H} \leq \mathcal{G}_K$, such that $\mathcal{G}_{K_{\mathrm{tr}}}\mathcal{H} = \mathcal{G}_K$, $\mathcal{G}_{K_{\mathrm{tr}}} \cap \mathcal{H} = \{1\}$ and $\mathcal{H} \cong \mathcal{G}(K_{\mathrm{tr}}/K)$. In view of Galois theory and the Mel'nikov-Tavgen' theorem [28], these results imply in the case of $\mathrm{char}(\widehat{K}) = q > 0$ the existence of a field $K' \in I(K_{\mathrm{sep}}/K)$ satisfying the following conditions:

(3.4) $K' \cap K_{\mathrm{tr}} = K$, $K'K_{\mathrm{tr}} = K_{\mathrm{sep}}$ and $K_{\mathrm{sep}} \cong K_{\mathrm{tr}} \otimes_K K'$ over K; the field $\widehat{K}'$ is a perfect closure of $\widehat{K}$, finite extensions of K in K' are of q-primary degrees, $K_{\mathrm{sep}} = K'_{\mathrm{tr}}$, $v(K') = qv(K')$, and the natural embedding of K into K' induces isomorphisms $v(K)/pv(K) \cong v(K')/pv(K')$, $p \in \mathbb{P} \setminus \{q\}$.

Assume as above that (K, v) is Henselian. Then each $\Delta \in d(K)$ has a unique, up-to an equivalence, valuation v_Δ extending v so that the value group $v(\Delta)$ of (Δ, v_Δ) is totally ordered and abelian (cf. [38], Ch. 2, Sect. 7). It is known that $v(K)$ is a subgroup of $v(\Delta)$ of index $e(\Delta/K) \leq [\Delta\colon K]$, and the residue division ring $\widehat{\Delta}$ of (Δ, v_Δ) is a $\widehat{K}$-algebra. Moreover, by the Ostrowski-Draxl theorem [10], $[\Delta\colon K]$ is divisible by $e(\Delta/K)[\widehat{\Delta}\colon \widehat{K}]$, and in case $\mathrm{char}(\widehat{K}) \dagger [\Delta\colon K]$, $[\Delta\colon K] = e(\Delta/K)[\widehat{\Delta}\colon \widehat{K}]$. An algebra $D \in d(K)$ is called inertial, if $[D\colon K] = [\widehat{D}\colon \widehat{K}]$ and $\widehat{D} \in d(\widehat{K})$. Inertial K-algebras and algebras from $d(\widehat{K})$ are related as follows (see [18], Theorem 2.8):

(3.5) (a) Each $\widetilde{D} \in d(\widehat{K})$ has an inertial lift over K, i.e. $\widetilde{D} = \widehat{D}$, for some $D \in d(K)$ inertial over K, that is uniquely determined by $\widetilde{D}$, up-to a K-isomorphism.

(b) The set $\mathrm{IBr}(K) = \{[I] \in \mathrm{Br}(K)\colon\ I \in d(K) \text{ is inertial}\}$ is a subgroup of $\mathrm{Br}(K)$; the canonical map $\mathrm{IBr}(K) \to \mathrm{Br}(\widehat{K})$ is an index-preserving isomorphism.

4 Proof of Theorem 2.1 (a) and (c)

The role of Lemma 3.1 in the study of Brauer p-dimensions of FG-extensions of a field E is determined by the following result of [7], which characterizes the condition $\mathrm{abrd}_p(E) \leq \mu$, for a given $\mu \in \mathbb{N}$. When E is virtually perfect, this result is in fact equivalent to [33], Lemma 1.1, and in case $\mu = 1$, it restates Theorem 3 of [1], Ch. XI.

Lemma 4.1. *Let E be a field, $p \in \mathbb{P}$ and $\mu \in \mathbb{N}$. Then $\mathrm{abrd}_p(E) \leq \mu$ if and only if, for each $E' \in \mathrm{Fe}(E)$, $\mathrm{ind}(\Delta) \leq p^{\mu}$ whenever $\Delta \in d(E')$ and $\exp(\Delta) = p$. Moreover, if E is virtually perfect, then $\mathrm{abrd}_p(E) \geq \mathrm{Brd}_p(E')$, for all finite extensions E'/E.*

Let now F/E be a transcendental FG-extension and $F_0 \in I(F/E)$ a rational extension of E with $\mathrm{trd}(F_0/E) = \mathrm{trd}(F/E) = t$. Clearly, an ordering on a fixed transcendency basis of F_0/E gives rise to a height t E-valuation v_0 of F_0 with $v_0(F_0) = \mathbb{Z}^t$ and $\widehat{F}_0 = E$. Considering any prolongation of v_0 on F, and taking into account that $[F\colon F_0] < \infty$, one obtains the following:

(4.1) F has an E-valuation v of height t, such that $v(F) \cong \mathbb{Z}^t$ and $\widehat{F}$ is a finite extension of E; in particular, $v(F)/pv(F)$ is a group of order p^t, for every $p \in \mathbb{P}$.

When $\mathrm{char}(E) = p$, (4.1) implies $[\widehat{F}\colon \widehat{F}^p] = [E\colon E^p]$ (cf. [23], Ch. VII, Sect. 7), so the former assertion of Theorem 2.1 (c) follows from the next lemma.

Lemma 4.2. *Let (K, v) be a valued field with $\mathrm{char}(K) = q > 0$ and $v(K) \neq qv(K)$, and let $\tau(q)$ be the dimension of $v(K)/qv(K)$ as a vector space over the field $\mathbb{F}_q$ with q elements. Then:*

(a) *For each $\pi \in K^*$ with $v(\pi) \notin qv(K)$, there are degree q extensions L_m of K in $K(q)$, $m \in \mathbb{N}$, such that the compositum $M_m = L_1 \dots L_m$ has a unique valuation v_m extending v, up-to an equivalence, $(M_m, v_m)/(K, v)$ is totally ramified, $[M_m\colon K] = q^m$ and $v(\pi) \in q^m v_m(M_m)$, for each m;*

(b) *Given an integer $n \geq 2$, there exists $T_n \in d(K)$ with $\exp(T_n) = q$ and $\mathrm{ind}(T_n) = q^{n-1}$ except, possibly, if $\tau(q) < \infty$ and $[\widehat{K}\colon \widehat{K}^q] < q^{n-\tau(q)}$.*

Proof. It suffices to consider the special case of $v(\pi) < 0$. Fix a Henselization $(K_v, \bar{v})$ of (K, v), put $\rho(K_v) = \{u^q - u\colon\ u \in K_v\}$, and for each $m \in \mathbb{N}$, denote by L_m the root field in K_{sep} over K of the polynomial $f_m(X) = X^q - X - \pi_m$, where $\pi_m = \pi^{1+qm}$. Also, let $\mathbb{F}$ be the prime subfield of K, $\Phi = \mathbb{F}(\pi)$, ω the valuation of Φ induced by v, and $(\Phi_\omega, \bar{\omega})$ a Henselization of (Φ, ω), such that $\Phi_\omega \subseteq K_v$ and $\bar{v}$ extends $\bar{\omega}$ (the existence of $(\Phi_\omega, \bar{\omega})$ follows from [13], Theorem 15.3.5). Identifying K_v with its K-isomorphic copy in K_{sep}, put $L'_m = L_m K_v$ and $M'_m = M_m K_v$, for every index m. It is easily verified that $\rho(K_v)$ is an $\mathbb{F}$-subspace of K_v and $\bar{v}(u^q - u) \in q\bar{v}(K_v)$, for every $u \in K_v$ with $\bar{v}(u) < 0$. As $\bar{v}(K_v) = v(K)$, this observation and the choice of π indicate that the cosets $\pi_m + \rho(K_v)$, $m \in \mathbb{N}$, are

linearly independent over $\mathbb{F}$. In view of the Artin-Schreier theorem and Galois theory (cf. [23], Ch. VIII, Sect. 6), this implies $f_m(X)$ is irreducible over K_v, L'_m/K_v and L_m/K are cyclic extensions of degree q, M'_m/K_v and M_m/K are abelian, and $[M'_m : K_v] = [M_m : K] = q^m$, for each $m \in \mathbb{N}$. Moreover, our argument proves that degree q extensions of K_v in the compositum of the fields L'_m, $m \in \mathbb{N}$, are cyclic and totally ramified over K_v. At the same time, it follows from the Henselity of $\bar{v}$ and the equality $\widehat{K}_v = \widehat{K}$ that M'_m contains as a subfield an inertial lift over K_v of the separable closure of $\widehat{K}$ in $\widehat{M'_m}$. When v is discrete and $\widehat{K}$ is perfect, the obtained results imply the assertions of Lemma 4.2 (a), since finite extensions of K_v in $K_{\rm sep}$ are defectless (relative to $\bar{v}$, see [23], Ch. XII, Sect. 6, Corollary 2).

To prove Lemma 4.2 (a) in general it remains to be seen that, for any fixed $m \in \mathbb{N}$, M_m has a unique, up-to an equivalence, valuation v_m extending v, $(M_m, v_m)/(K, v)$ is totally ramified and $v(\pi) \in q^m v(M_m)$. The extendability of v to a valuation v_m of M_m is well-known (cf. [23], Ch. XII, Sect. 4), so our assertions can be deduced from the concluding one, the equality $[M_m : K] = [M_m K_v : K_v] = q^m$ and statement (3.1). Our proof also relies on the fact that (Φ, ω) is a discrete valued field and $\widehat{\Phi}/\mathbb{F}$ is a finite extension (see [3], Ch. II, Lemma 3.1, or [13], Example 4.1.3); in particular, $\widehat{\Phi}$ is perfect. Let now $\Psi_m \in I(K_{\rm sep}/\Phi)$ be the root field of $f_m(X)$ over Φ. Then $L_m = \Psi_m K$, $[\Psi_m : \Phi] = q$, $M_m = \Theta_m K$ and $[\Theta_m : \Phi] = q^m$, where $\Theta_m = \Psi_1 \dots \Psi_m$. Therefore, $\Theta_m \Phi_\omega/\Phi_\omega$ is totally ramified relative to $\bar{\omega}$. Equivalently, the integral closure of $O_\omega(\Phi)$ in Θ_m contains a primitive element t'_m of Θ_m/Φ, whose minimal polynomial $\theta_m(X)$ over $O_\omega(\Phi)$ is Eisensteinian (cf. [3], Ch. I, Theorem 6.1, and [23], Ch. XII, Sects. 2, 3 and 6). Hence, ω has a unique prolongation ω_m on Θ_m, up-to an equivalence, $\omega(t_m) \notin q\omega(\Phi)$ and $q^m \omega_m(t'_m) = \omega(t_m)$, where t_m is the free term of $\theta_m(X)$. As $\pi \in \Phi$, $v(\pi) \notin qv(K)$ and Θ_m/Φ is a Galois extension, this implies t'_m is a primitive element of M_m/K and M'_m/K_v, $q^m v_m(t'_m) = v(t_m) = \omega(t_m)$ and $v(\pi) \in q^m v_m(M_m)$, which completes the proof of Lemma 4.2 (a).

We prove Lemma 4.2 (b). Put $\pi_1 = \pi$ and suppose that there exist elements $\pi_j \in K^*$, $j = 2, \dots, n$, and an integer $\mu \le n$, such that the cosets $v(\pi_i) + qv(K)$, $i = 1, \dots, \mu$, are linearly independent over $\mathbb{F}_q$, and in case $\mu < n$, $v(\pi_u) = 0$ and the residue classes $\hat{\pi}_u$, $u = \mu + 1, \dots, n$, generate an extension of $\widehat{K}^q$ of degree $q^{n-\mu}$. Fix a generator λ_m of $\mathcal{G}(L_m/K)$, for each $m \in \mathbb{N}$, denote by T_n the K-algebra $\otimes_{j=2}^n (L_{j-1}/K, \lambda_{j-1}, \pi_j)$, where $\otimes = \otimes_K$, and put $T'_n = T_n \otimes_K K_v$. We show that $T_n \in d(K)$ (whence $\exp(T_n) = q$ and $\mathrm{ind}(T_n) = q^{n-1}$). Clearly, there is a K_v-isomorphism $T'_n \cong \otimes_{j=2}^n (L'_{j-1}/K_v, \lambda'_{j-1}, \pi_j)$, where $\otimes = \otimes_{K_v}$ and λ'_{j-1} is the unique K_v-automorphism of L'_{j-1} extending λ_{j-1}, for each j. Therefore, it suffices for the proof of Lemma 4.2 (b) to show that $T'_n \in d(K_v)$. Since K_v and L'_m, $m \in \mathbb{N}$, are related as K and L_m, $m \in \mathbb{N}$, this amounts to proving that $T_n \in d(K)$, for (K, v) Henselian. Suppose first that $n = 2$. As L_1/K is totally ramified, it follows from the Henselity of v that $v(l) \in qv(L_1)$, for every element l of the norm group $N(L_1/K)$. One also concludes that if $l \in N(L_1/K)$ and $v_L(l) = 0$, then $\hat{l} \in \widehat{K}^q$. These observations prove that $\pi_2 \notin N(L_1/K)$, so it follows from [34], Sect. 15.1, Proposition b, that $T_2 \in d(K)$. Henceforth, we assume that $n \ge 3$ and view all value groups considered in the rest of the proof as (ordered) subgroups of a fixed divisible hull of $v(K)$. Note that the centralizer C_n of L_n in T_n is L_n-isomorphic to $T_{n-1} \otimes_K L_n$ and $\otimes_{j=2}^{n-1} (L_{j-1} L_n, \lambda_{j-1,n}, \pi_j)$, where $\otimes = \otimes_{L_n}$ and $\lambda_{j-1,n}$ is the unique L_n-automorphism of $L_{j-1} L_n$ extending

λ_{j-1}, for each index j. Therefore, using (3.1) and Lemma 4.2 (a), one obtains inductively that it suffices to prove that $T_n \in d(K)$, provided $C_n \in d(L_n)$.

Denote by w_n the valuation of C_n extending v_{L_n}, and by $\widehat{C}_n$ its residue division ring. It follows from the Ostrowski-Draxl theorem that $w_n(C_n)$ equals the sum of $v(M_n)$ and the group generated by $q^{-1}v(\pi_{i'})$, $i' = 2, \dots, n-1$. Similarly, it is proved that $\widehat{C}_n$ is a field and $\widehat{C}_n^q \subseteq \widehat{K}$. One also sees that $\widehat{C}_n \neq \widehat{K}$ if and only if $\mu < n-1$, and in this case, $[\widehat{C}_n \colon \widehat{K}] = q^{n-1-\mu}$ and $\hat{\pi}_u \in \widehat{C}_n^q$, $u = \mu+1, \dots, n-1$. These results show that $v(\pi_n) \notin qw_n(C_n)$, if $\mu = n$, and $\hat{\pi}_n \notin \widehat{C}_n^q$ when $\mu < n$. Let now $\bar{\lambda}_n$ be the K-automorphism of C_n extending both λ_n and the identity of the natural K-isomorphic copy of T_{n-1} in C_n, and let $t'_n = \prod_{\kappa=0}^{q-1} \bar{\lambda}_n^{\kappa}(t_n)$, for each $t_n \in C_n$. Then, by Skolem-Noether's theorem (cf. [34], Sect. 12.6), $\bar{\lambda}_n$ is induced by an inner K-automorphism of T_n. This implies $w_n(t_n) = w_n(\bar{\lambda}_n(t_n))$ and $w_n(t'_n) \in qw_n(C_n)$, for all $t_n \in C_n$, and yields $\hat{t}'_n \in \widehat{C}_n^q$ when $w_n(t_n) = 0$. Therefore, $t'_n \neq \pi_n$, $t_n \in C_n$, so it follows from [1], Ch. XI, Theorems 11 and 12, that $T_n \in d(K)$. Lemma 4.2 is proved. □

Proof of the latter assertion of Theorem 2.1 (c). Assume that F/E is an FG-extension, such that $\mathrm{char}(E) = p$, $[E \colon E^p] = p^\nu < \infty$ and $\mathrm{trd}(F/E) = t \geq 1$. This implies $[F \colon F^p] = p^{\nu+t}$, so it follows from Lemma 4.1 and [1], Ch. VII, Theorem 28, that $\mathrm{Brd}_p(F) \leq \mathrm{abrd}_p(F) \leq \nu + t$. At the same time, it is clear from (4.1) and Lemma 4.2 that there exists $\Delta \in d(F)$ with $\exp(\Delta) = p$ and $\mathrm{ind}(\Delta) = p^{\nu+t-1}$, which yields $\mathrm{Brd}_p(F) \geq \nu + t - 1$ and so completes our proof.

Our next lemma is implied by (3.5), Lemma 3.1 and the immediacy of Henselizations of valued fields (cf. [13], Theorems 15.2.2 and 15.3.5).

Lemma 4.3. *Let E be a field, $F = E(X)$ a rational extension of E with $\mathrm{trd}(F/E) = 1$, $f(X) \in E[X]$ an irreducible polynomial over E, M an extension of E generated by a root of f in E_{sep}, v a discrete E-valuation of F with a uniform element f, and $(F_v, \bar{v})$ a Henselization of (F, v). Also, let $\widetilde{D} \in d(M)$ be an algebra of exponent $p \in \mathbb{P}$. Then M is E-isomorphic to the residue field of (F, v) and $(F_v, \bar{v})$, and there exists $D \in d(F)$ with $\exp(D) = p$ and $[D \otimes_F F_v] = [D']$, where $D' \in d(F_v)$ is an inertial lift of $\widetilde{D}$ over F_v.*

Proof of Theorem 2.1 (a). Let $\mathrm{abrd}_p(E) = \lambda \in \mathbb{N}$ and $F = E(X_1, \dots, X_\kappa)$. Then, by Lemma 4.1, there exists $M \in \mathrm{Fe}(E)$, such that $d(M)$ contains an algebra $\widetilde{\Delta}$ with $\exp(\widetilde{\Delta}) = p$ and $\mathrm{ind}(\widetilde{\Delta}) = p^\lambda$. We show that there is $\Delta \in d(F)$ with $\exp(\Delta) = p$ and $\mathrm{ind}(\Delta) \geq p^{\lambda+\kappa-1}$. Suppose first that $\kappa = 1$, take a primitive element α of M/E, and denote by $f(X_1)$ its minimal monic polynomial over E. Attach to f a discrete valuation v of F and fix $(F_v, \bar{v})$ as in Lemma 4.3. Then, by Lemma 3.1, there is $\Delta_1 \in d(F)$ with $[\Delta_1 \otimes_F F_v] = [\overline{\Delta}]$ and $\exp(\Delta_1) = p$, where $\overline{\Delta}$ is an inertial lift of $\widetilde{\Delta}$ over F_v. Since $\overline{\Delta} \in d(F_v)$ and $\mathrm{ind}(\overline{\Delta}) = p^\lambda$, this indicates that $p^\lambda \mid \mathrm{ind}(\Delta_1)$, which proves Theorem 2.1 (a) when $\kappa = 1$. In addition, Lemma 3.2 implies that there exist infinitely many degree p cyclic extensions of F in F_v. Hence, F_v contains as a subfield a Galois extension R_κ of F with $\mathcal{G}(R_\kappa/F)$ of order $p^{\kappa-1}$ and period p. When $\mathrm{ind}(\Delta_1) = p^\lambda$, this makes it easy to deduce the existence of Δ, for an arbitrary κ, from (4.1) (with a ground field $E(X_1)$ instead of E) and [31], Theorem 1, or else, by repeatedly using the Proposition in [34], Sect. 19.6. It remains to consider the case where $\kappa \geq 2$ and

there exists $D_1 \in d(E(X_1))$ with $\exp(D_1) = p$ and $\mathrm{ind}(D_1) = p^{\lambda'} > p^{\lambda}$. It is easily verified that $D_1 \otimes_{E(X_1)} E(X_1)((X_2)) \in d(E(X_1)((X_2)))$, and it follows from Lemma 3.2 that there are infinitely many degree p cyclic extensions of $E(X_1, X_2)$ in $E(X_1)((X_2))$. As in the case of $\kappa = 1$, this enables one to prove the existence of $\Delta' \in d(F)$ with $\exp(\Delta') = p$ and $\mathrm{ind}(\Delta') = p^{\lambda'+\kappa-2} \geq p^{\lambda+\kappa-1}$. Thus Theorem 2.1 (a) is proved.

Corollary 4.4. *Let E be a field and F/E a rational extension with* $\mathrm{trd}(F/E) = \infty$. *Then* $\mathrm{Brd}_p(F) = \infty$, *for every $p \in \mathbb{P}$.*

Proof. This follows from Theorem 2.1 (a) and the fact that, for any rational field extension F'/F with $\mathrm{trd}(F'/F) = 2$, there is an E-isomorphism $F \cong F'$, whence $\mathrm{Brd}_p(F) = \mathrm{Brd}_p(F')$, for each $p \in \mathbb{P}$. □

Remark 4.5. *Let E be a field with $abrd_p(E) = \infty$, $p \in \mathbb{P}$, and let F/E be a transcendental FG-extension. Then it follows from (1.1) (b), (c) and Theorem 2.1 (b) that Brauer pairs $(m, n) \in \mathbb{N}^2$ are index-exponent pairs over F. Therefore, Corollary 4.4 with its proof implies the latter assertion of (1.2).*

Alternatively, it follows from Galois theory, Lemmas 3.2, 4.3 and basic theory of valuation prolongations that $r_p(\Phi) = \infty$, $p \in \mathbb{P}$, for every transcendental FG-extension Φ/E. Hence, by [11] and Witt's lemma (cf. [9], Sect. 15, Lemma 2), finite abelian groups are realizable as Galois groups over Φ, so both parts of (1.2) can be proved by the method used in [34], Sect. 19.6.

Proposition 4.6. *Let F/E be an* FG*-extension with* $\mathrm{trd}(F/E) = t \geq 1$ *and* $\mathrm{abrd}_p(E) < \infty$, *$p \in P$, for some subset $P \subseteq \mathbb{P}$. Then P possesses a finite subset $P(F/E)$, such that* $\mathrm{Brd}_p(F) \geq \mathrm{abrd}_p(E) + t - 1$, *$p \in P \setminus P(F/E)$.*

Proof. It follows from (1.1) (c) and Theorem 2.1 (a) that one may take as $P(F/E)$ the set of divisors of $[F\colon F_0]$ lying in P, for some rational extension F_0 of E in F with $\mathrm{trd}(F_0/E) = t$.

Example 4.7. *There exist field extensions F/E satisfying the conditions of Proposition 4.6, for $P = \mathbb{P}$, such that $P(F/E)$ is nonempty. For instance, let E be a real closed field, Φ the function field of the Brauer-Severi variety attached to the symbol E-algebra $A = A_{-1}(-1, -1; E)$, and F/Φ a finite field extension with $\sqrt{-1} \notin F$. Then $abrd(F) = 0 <$* $\mathrm{abrd}_2(E) = 1$ *(see the example in [6]) and $abrd_p(E) = 0$, $p > 2$, which implies $P(F/E) = \{2\}$ and $P = \mathbb{P}$.*

5 Proof of Theorem 2.1 (b)

The former claim of Theorem 2.1 (b) is implied by the following lemma.

Lemma 5.1. *Let K be a field with* $\mathrm{abrd}_p(K) = \infty$, *for some $p \in \mathbb{P}$, and let F/K be an* FG*-extension with* $\mathrm{trd}(F/K) \geq 1$. *Then there exist $D_\nu \in d(F)$, $\nu \in \mathbb{N}$, such that* $\exp(D_\nu) = p$ *and* $\mathrm{ind}(D_\nu) \geq p^\nu$.

Proof. Statement (1.1) (c) implies the class of fields Φ with $\mathrm{abrd}_p(\Phi) = \infty$ is closed under the formation of finite extensions. Since K has a rational extension F_0 in F with $\mathrm{trd}(F_0/K) = \mathrm{trd}(F/K)$, whence $[F\colon F_0] < \infty$, this shows that it is sufficient to prove Lemma 5.1 in the case of $F = F_0$. Note also that $\mathrm{ind}(T_0 \otimes_K F_0) = \mathrm{ind}(T_0)$ and $\exp(T_0 \otimes_K F_0) = \exp(T_0)$, for each $T_0 \in d(K)$, so one may assume, for the proof, that $F = F_0$ and $\mathrm{trd}(F/K) = 1$. It follows from Lemma 4.1 and the equality $\mathrm{abrd}_p(K) = \infty$ that there are $M_\nu \in \mathrm{Fe}(K)$ and $\widetilde{D}_\nu \in d(M_\nu)$, $\nu \in \mathbb{N}$, with $\exp(\widetilde{D}_\nu) = p$ and $\mathrm{ind}(\widetilde{D}_\nu) \geq p^\nu$, for each index ν. Hence, by Lemmas 4.3 and 3.1, there exist a discrete K-valuation v_ν of F, and an algebra $D_\nu \in d(F)$, such that the residue field of (F, v_ν) is K-isomorphic to M_ν, $\exp(D_\nu) = p$, and $[D_\nu \otimes_F F_{v_\nu}] = [D'_\nu]$, where D'_ν is an inertial lift of $\widetilde{D}_\nu$ over F_{v_ν}. This implies $\mathrm{ind}(\widetilde{D}_\nu) \mid \mathrm{ind}(D_\nu)$, $\nu \in \mathbb{N}$, proving Lemma 5.1. □

To prove the latter part of Theorem 2.1 (b) we need the following lemma.

Lemma 5.2. *Let A, B and C be algebras over a field F, such that $A, B, C \in s(F)$, $A = B \otimes_F C$, $\exp(C) = p \in \mathbb{P}$, and $\exp(B) = \mathrm{ind}(B) = p^m$, for some $m \in \mathbb{N}$. Assume that $\mathrm{ind}(A) = p^n > p^m$ and k is an integer with $m < k \leq n$. Then there exists $T_k \in s(F)$ with $\exp(T_k) = p^m$ and $\mathrm{ind}(T_k) = p^k$.*

Proof. When $k = n$, there is nothing to prove, so we assume that $k < n$. By [29], Sect. 4, Theorem 2, $[C] = [\Delta_1 \otimes_F \dots \otimes_F \Delta_\nu]$, where $\nu \in \mathbb{N}$ and for each index j, $\Delta_j \in d(F)$ and $\mathrm{ind}(\Delta_j) = p$. Put $T_j = B \otimes_F (\Delta_1 \otimes_F \dots \otimes_F \Delta_j)$ and $t_j = \deg(T_j)/\mathrm{ind}(T_j)$, $j = 1, \dots, \nu$, and let $S(A)$ be the set of those j, for which $\mathrm{ind}(T_j) \geq p^k$. Clearly, $S(A) \neq \phi$ and the set $S_0(A) = \{i \in S(A)\colon\ t_i \leq t_j, j \in S(A)\}$ contains a minimal index γ. The conditions of Lemma 5.2 ensure that $\exp(T_j) = p^m$, so $\mathrm{ind}(T_j) = p^{m(j)}$, where $m(j) \in \mathbb{N}$, for each $j \in S(A)$. We show that $\mathrm{ind}(T_\gamma) = p^k$. If $\gamma = 1$, then (1.1) (c) and the inequality $m < k$ imply $k = m + 1$ and $\mathrm{ind}(T_1) = p^k$, as claimed. Suppose now that $\gamma \geq 2$. Then it follows from (1.1) (b) that $\mathrm{ind}(T_\gamma) = \mathrm{ind}(T_{\gamma-1}).p^\mu$, for some $\mu \in \{-1, 0, 1\}$. The possibility that $\mu \neq 1$ is ruled out, since it contradicts the fact that $\gamma \in S_0(A)$. This yields $\mathrm{ind}(T_\gamma) = \mathrm{ind}(T_{\gamma-1}).p$ and $t_\gamma = t_{\gamma-1}$. As γ is minimal in $S_0(A)$, it is now easy to see that $\mathrm{ind}(T_{\gamma-u}) = p^{k-u}$, $u = 0, 1$, which proves Lemma 5.2. □

The conditions of Lemma 5.2 are fulfilled, for each $m \in \mathbb{N}$ and infinitely many integers $n > m$, if $\mathrm{char}(E) = p$, E is *not* virtually perfect and F/E satisfies the conditions of Theorem 2.1. Since, by Witt's lemma, cyclic p-extensions of F are realizable as intermediate fields of $\mathbb{Z}_p$-extensions of F, this can be obtained by applying (1.1) (b), (4.1) and Lemma 4.2 together with general properties of cyclic F-algebras, see [34], Sect. 15.1, Corollary b and Proposition b. Thus Theorem 2.1 is proved in the case of $p = \mathrm{char}(E)$. For the proof of the latter assertion of Theorem 2.1 (b), when $p \neq \mathrm{char}(E)$, we need the following lemma.

Lemma 5.3. *Let K be a field and F/K an* FG*-extension with* $\mathrm{trd}(F/K) = 1$. *Then, for each $p \in \mathbb{P}$ different from char(K), there exist non-equivalent discrete K-valuations v_m of F, $m \in \mathbb{N}$, satisfying the following:*

(a) *For any $m \in \mathbb{N}$, (F, v_m) possesses a totally ramified extension (F_m, w_m), such that $F_m \in I(F_{\mathrm{sep}}/F)$, F_m/F is cyclic and $[F_m\colon F] = p^m$;*

(b) *The valued fields (F_m, w_m) can be chosen so that $F_{m'} \cap F_{\bar{m}} = F$, $m' \neq \bar{m}$.*

Proof. Let $X \in F$ be a transcendental element over K. Then $F/K(X)$ is a finite extension, and the separable closure of $K(X)$ in F is unramified relative to every discrete K-valuation of $K(X)$, with at most finitely many exceptions (up-to an equivalence, see [3], Ch. I, Sect. 5). This reduces the proof of Lemma 5.3 to the special case of $F = K(X)$. For each $m \in \mathbb{N}$, let $\delta_m \in F_{\rm sep}$ be a primitive p^m-th root of unity, $K_m = K(\delta_m)$, $f_m(X) \in K[X]$ the minimal polynomial of δ_m over K, and ρ_m a discrete K-valuation of F with a uniform element f_m. Clearly, the valuations ρ_m, $m \in \mathbb{N}$, are pairwise non-equivalent. Also, it is well-known (see [23], Ch. V, Theorem 6; Ch. VIII, Sect. 3, and [17], Ch. 4, Sect. 1) that if $m', \bar{m} \in \mathbb{N}$, then the extension $K_{m'}(\delta_{\bar{m}})/K_{m'}$ are cyclic except, possibly, in the case where $m' = 1$, $\bar{m} > 2$, $p = 2$, $\mathrm{char}(K) = 0$ and $\delta_2 \notin K$. Denote by v_m the valuation ρ_{m+1}, for each m, if $p = 2$, $\mathrm{char}(K) = 0$ and $\delta_2 \notin K$, and put $v_m = \rho_m$, $m \in \mathbb{N}$, otherwise. Since $p \neq \mathrm{char}(K)$, and by Lemma 4.5, K_m is K-isomorphic to the residue field of (F, ρ_m), we have $\delta_m \in F_{v_m}$, where F_{v_m} is a Henselization of F in $F_{\rm sep}$ relative to v_m. This enables one to deduce from Kummer theory that F_{v_m} possesses a totally ramified cyclic extension L_{v_m} of degree p^m. Furthermore, it follows from the choice of v_m and the observation on the extensions $K_{m'}(\delta_{\bar{m}})/K_{m'}$ that $F_{v_{m'}}(\delta_{\bar{m}})/F_{v_{m'}}$ are cyclic, for all pairs $m', \bar{m} \in \mathbb{N}$. Hence, by the generalized Grunwald-Wang theorem (cf. [26], Theorems 1 (ii) and 2) and the note preceding the statement of Lemma 3.2, there exist totally ramified extensions $(F_m, w_m)/(F, v_m)$, $m \in \mathbb{N}$, such that $F_m \in I(F_{\rm sep}/F)$, F_m/F is cyclic with $[F_m \colon F] = p^m$, for each m, and in case $m \geq 2$, F_m/F is unramified relative to $v_1, \dots, v_{m-1}$. This ensures that $F_{m'} \cap F_{\bar{m}} = F$, $m' \neq \bar{m}$, and so completes the proof of Lemma 5.3. $\square$

Proof of the latter statement of Theorem 2.1 (b). Let $\mathrm{abrd}_p(E) = \infty$, for some $p \in \mathbb{P}$. In view of (1.1) (b), Lemmas 3.1, 5.1 and 5.2, it is sufficient to show that there exists $A_m \in d(F)$ with $\exp(A_m) = \mathrm{ind}(A_m) = p^m$, for any fixed $m \in \mathbb{N}$. As in the proof of Lemma 5.1, our considerations reduce to the special case of $\mathrm{trd}(F/K) = 1$. Analyzing this proof, one obtains that there is $M \in \mathrm{Fe}(E)$, such that $d(M)$ contains a cyclic M-algebra $\widetilde{A}_1$ of degree p, and when $p \neq \mathrm{char}(E)$, M contains a primitive p^m-th root of unity δ_m. Note further that M can be chosen so as to be E-isomorphic to the residue field $\widehat{F}$ of F relative to some discrete E-valuation v. In view of Kummer theory (see [23], Ch. VIII, Sect. 6) and Witt's lemma, the assumptions on M ensure that each degree p cyclic extension Y_1 of M lies in $I(Y_m/M)$, for some degree p^m cyclic extension Y_m/M. Suppose now that Y_1 embeds in $\widetilde{A}_1$ as an M-subalgebra, fix a generator τ_1 of $\mathcal{G}(Y_1/M)$ and an automorphism τ_m of Y_m extending τ_1. Then $\widetilde{A}_1$ is isomorphic to the cyclic M-algebra $(Y_1/M, \tau_1, \tilde{\beta})$, for some $\tilde{\beta} \in M^*$, τ_m generates $\mathcal{G}(Y_m/M)$, the M-algebra $\widetilde{A}_m = (Y_m/M, \tau_m, \tilde{\beta})$ lies in $s(M)$, and we have $p^{m-1}[\widetilde{A}_m] = [\widetilde{A}_1]$ (cf. [34], Sect. 15.1, Corollary b). Therefore, $\widetilde{A}_m \in d(M)$ and $\mathrm{ind}(\widetilde{A}_m) = \exp(\widetilde{A}_m) = p^m$. Assume now that (F, v) has a valued extension (L, v_L), such that L/F is cyclic, $[L \colon F] = p^m$ and the residue field of (L, v_L) is E-isomorphic to Y_m. Then $\mathcal{G}(L/F) \cong \mathcal{G}(Y_m/M)$, and for each generator σ of $\mathcal{G}(L/F)$ and pre-image β of $\tilde{\beta}$ in $O_v(F)$, the algebra $A_m = (L/F, \sigma, \beta)$ lies in $d(F)$ (see [34], Sect. 15.1, Proposition b, and [18], Theorem 5.6). Note also that $\mathrm{ind}(A_m) = \exp(A_m) = p^m$ and σ can be chosen so that $A_m \otimes_F F_v$ be an inertial lift of $\widetilde{A}_m$ over F_v. When $p > 2$, this completes the proof of Theorem 2.1 (b), since Lemma 3.2 guarantees in this case the existence of a valued extension

(L, v_L) of (F, v) with the above-noted properties.

Similarly, one concludes that if $p = 2$, then it suffices to prove Theorem 2.1 (b), provided $\mathrm{char}(E) = 0$ and $\mathcal{G}(E(\delta_m)/E)$ is noncyclic, where δ_m is a primitive 2^m-th root of unity in E_{sep}. This implies the group $E_1^*/E_1^{*2^\nu}$ has period 2^ν, for each $\nu \in \mathbb{N}$, $E_1 \in \mathrm{Fe}(E)$ (cf. [23], Ch. VIII, Sects. 3 and 9). Take a valued extension $(F_m, w_m)/(F, v_m)$ as required by Lemma 5.3, and denote by $\widehat{F}_m$ the residue field of (F, v_m). Fix a generator ψ_m of $\mathcal{G}(F_m/F)$ and an element $\tilde{\beta}_m \in \widehat{F}_m^*$ so that $\tilde{\beta}_m^{2^{m-1}} \notin \widehat{F}_m^{*2^m}$, and put $A_m = (F_m/F, \psi_m, \beta_m)$, for some pre-image β_m of $\tilde{\beta}_m$ in $O_{v_m}(F)$. As $(F_m, w_m)/(F, v_m)$ is totally ramified, w_m is uniquely determined by v_m, up-to an equivalence. Therefore, $w_m(\lambda_m) = w_m(\psi_m(\lambda_m))$, for all $\lambda_m \in F_m$, and when $w_m(\lambda_m) = 0$, $\widehat{F}_m^{*2^m}$ contains the residue class of the norm $N_F^{F_m}(\lambda_m)$. Now it follows from [34], Sect. 15.1, Proposition b, that $A_m \in d(F)$ and $\mathrm{ind}(A_m) = \exp(A_m) = 2^m$, so Theorem 2.1 is proved.

Corollary 5.4. *Let E be a field with* $\mathrm{abrd}(E) = \infty$. *Then* $\mathrm{Brd}(F) = \infty$, *for every transcendental* FG-*extension* F/E.

Proof. The equality $\mathrm{abrd}(E) = \infty$ means that either $\mathrm{abrd}_{p'}(E) = \infty$, for some $p' \in \mathbb{P}$, or $\mathrm{abrd}_p(E)$, $p \in \mathbb{P}$, is an unbounded number sequence. In view of Theorem 2.1 (b) and Proposition 4.6, this proves our assertion. □

Corollary 5.4 shows that a field E satisfies $\mathrm{abrd}(E) < \infty$, if its FG-extensions are of finite dimensions, in the sense of [2], Sect. 4. In view of (2.7) (a), this proves that Problem 4.4 of [2] is solved, generally, in the negative, even when all finite extensions of E have finite Brauer dimensions. Statements (2.7) also imply that both cases pointed out in the proof of Corollary 5.4 can be realized.

Remark 5.5. *Statement (2.6) indicates that if [2], Problem 4.5, is solved affirmatively in the class $\mathcal{A}$ of virtually perfect fields E with abrd$(E) < \infty$, then abrd$(E) \leq \dim(E)$. We show that such a solvability would imply the numbers $c(E)$, in (2.6), depend on the choice of E and may be arbitrarily large. Let C be an algebraically closed field, ν a positive integer and $C_\nu = C((X_1))\dots((X_\nu))$ the iterated formal Laurent formal power series field in ν variables over C. We prove that $c(C_\nu) \geq [\nu/2] - 1$. Note first that each FG-extension F/C_ν with trd$(F/C_\nu) = 1$ has a C-valuation f_ν, such that trd$(\widehat{F}/C) = 1$ and $f_\nu(F) = \mathbb{Z}^\nu$. Indeed, if $T \in F$ is a transcendental element over C_ν, $F_0 = C_\nu(T)$, and f_0 is the restricted Gauss valuation of F_0 extending the natural $\mathbb{Z}^\nu$-valued C-valuation of C_ν (see [13], Example 4.3.2), then one may take as f_ν any prolongation of f_0 on F. The equality trd$(\widehat{F}/C) = 1$ ensures that $r_p(\widehat{F}) = \infty$, for all $p \in \mathbb{P}$, which enables one to deduce from [31], Theorem 1, and [25], Corollary 1.4, that Brd$_p(F) = \mathrm{abrd}_p(F) = \nu$, $p \in \mathbb{P}$ and $p \neq \mathrm{char}(C)$ (see [25], page 37, for more details in case F/C_ν is rational). At the same time, it follows from [8], Proposition 7.1, that if char$(C) = 0$, then Brd$(C_\nu) = \mathrm{abrd}(C_\nu) = [\nu/2]$; hence, by (2.6), $c(C_\nu) \geq \mathrm{abrd}(F) - \mathrm{abrd}(C_\nu) - 1 = \nu - [\nu/2] - 1 \geq [\nu/2] - 1$, as claimed.*

Corollary 5.6. *Let F be a rational extension of an algebraically closed field F_0. Then* $\mathrm{trd}(F/F_0) = \infty$ *if and only if each Brauer pair $(m, n) \in \mathbb{N}^2$ is realizable as an index-exponent pair over F.*

Proof. If $\mathrm{trd}(F/F_0) = n < \infty$, then finite extensions of F are C_n-fields, by Lang-Tsen's theorem [22], so Lemma 4.1 and [27] imply $\mathrm{Brd}_p(F) < p^{n-1}$, $p \in \mathbb{P}$ (see [30], (16.10), for case $p = 2$). In view of (1.2), this completes our proof. $\square$

Theorem 2.1 and Example 4.7 lead naturally to the question of whether $\mathrm{Brd}_p(F) \geq k + \mathrm{trd}(F/E)$, provided that F/E is an FG-extension and $\mathrm{Brd}_p(E') = k < \infty$, $E' \in \mathrm{Fe}(E)$, for a given $p \in \mathbb{P}$. Our next result gives an affirmative answer to this question in several frequently used special cases:

Proposition 5.7. *Let E be a field and F an* FG*-extension of E with* $\mathrm{trd}(F/E) = n > 0$. *Suppose that there exists $M \in \mathrm{Fe}(E)$ satisfying the following condition, for some $p \in \mathbb{P}$ and $k \in \mathbb{N}$:*

(c) *For each $M' \in \mathrm{Fe}(M)$, there are $D' \in d(M')$ and $L' \in I(M'(p)/M')$, such that* $\exp(D') = [L' \colon M'] = p$, $\mathrm{ind}(D') = p^k$ *and* $D' \otimes_{M'} L' \in d(L')$.

Then there exist $D \in d(F)$, such that $\exp(D) = p$ *and* $\mathrm{ind}(D) \geq p^{k+n}$; *in particular,* $\mathrm{Brd}_p(F) \geq k + n$.

Proposition 5.7 is proved along the lines drawn in the proofs of Theorem 2.1 (a) and (b), so we omit the details. Note only that if $n \geq 2$ or $k = 1$, then D can be chosen so that $D \otimes_F F_v \in d(F_v)$, $[D \otimes_F F_v] \in \mathrm{Br}(F_{v,\mathrm{un}}/F_v)$ and $p^{n-1} \mid e(D \otimes_F F_v/F_v) \mid p^n$, for some E-valuation v of F with $\mathbb{Z}^{n-1} \leq v(F) \leq \mathbb{Z}^n$.

Remark 5.8. *Condition (c) of Proposition 5.7 is fulfilled, for $k = 1 = \mathrm{abrd}(E)$ and any $p \in \mathbb{P}$, if E is a global field or an FG-extension of an algebraically closed field E_0' with trd$(E/E_0') = 2$. It also holds when $k = 1$, $p \in \mathbb{P}$ and E is an FG-extension of a perfect PAC-field E_0 with trd$(E/E_0) = 1 = \mathrm{cd}_p(E_0)$ (see [12], Sect. 3, and [34], Sect. 19.3). In these cases, it can be deduced from (3.1) and [31], Theorem 1, that the power series fields $E_m = E((X_1))\dots((X_m))$, $m \in \mathbb{N}$, satisfy (c), for $k = 1 + m = \mathrm{abrd}_p(E_m)$ (cf. [25], Appendix A, or [8], (5.2) and Proposition 5.1). In addition, the conclusion of Proposition 5.7 is valid, if E is a local field, $k = 1$ and $p \in \mathbb{P}$, although (c) is then violated, for every p (see Proposition 6.3 with its proof, and appendices to [37] and [3], Ch. VI, Sect. 1).*

For a proof of the concluding result of this section, we refer the reader to [6]. When F/E is a rational extension and $r_p(E) \geq \mathrm{trd}(F/E)$, this result is contained in [32]. Combined with Lemma 3.2, it implies Nakayama's inequalities $\mathrm{Brd}_{p'}(F') \geq \mathrm{trd}(F'/E') - 1$, $p' \in \mathbb{P}$, for any FG-extension F'/E'.

Proposition 5.9. *Let F/E be an* FG*-extension with* $\mathrm{trd}(F/E) = n \geq 1$ *and* $\mathrm{cd}_p(\mathcal{G}_E) \neq 0$, *for some $p \in \mathbb{P}$. Then* $\mathrm{Brd}_p(F) \geq n$ *except, possibly, if $p = 2$, the Sylow pro-2-subgroups of $\mathcal{G}_E$ are of order* 2, *and F is a nonreal field.*

It is not known whether an FG-extension F/E with $\mathrm{trd}(F/E) = n \geq 3$ satisfies $\mathrm{abrd}_p(F) = \mathrm{Brd}_p(F) = n - 1$, provided that $p \in \mathbb{P}$, $\mathrm{cd}_p(\mathcal{G}_E) = 0$, and E is perfect in the case where $p = \mathrm{char}(E)$. It follows from (1.1) (c) that this question is equivalent to the Standard Conjecture on F/E (stated by Colliot-Thélène, see [25] and [24], Sect. 1) when E is algebraically closed. The question is also open in the case excluded by Proposition 5.9. Results like [27], Theorem 6.3 and Corollary 7.3, as well as statements (2.1) and (2.3)

attract interest in the problem of finding exact upper bounds on $\mathrm{abrd}_p(F)$, $p \in \mathbb{P}$. Specifically, it is worth noting that if E is algebraically closed and $\mathrm{Brd}_p(F) \geq p^{n-2}$, for infinitely many $p \in \mathbb{P}$, then this would solve negatively [2], Problem 4.5, by showing that $\mathrm{Brd}(F) = \infty$ whenever $n \geq 3$.

6 Reduction of (2.2) to the case of char$(E) = 0$

In this section we show that if $\mathcal{C}$ is a class of profinite groups and n is a positive integer, then the answer to (2.2) would be affirmative, for FG-extensions F/E with $\mathcal{G}_E \in \mathcal{C}$ and $\mathrm{trd}(F/E) \leq n$, if this holds when $\mathrm{char}(E) = 0$. This result can be viewed as a refinement of [13], Corollary 22.2.3, in the spirit of [24], 4.1.2.

Proposition 6.1. *Let E be a field of characteristic $q > 0$ and F/E an* FG-*extension. Then there exists an* FG-*extension L/E' satisfying the following:*

(a) $\mathrm{char}(E') = 0$, *$\mathcal{G}_{E'} \cong \mathcal{G}_E$ and* $\mathrm{trd}(L/E') = \mathrm{trd}(F/E)$*;*

(b) $\mathrm{Brd}_p(L) \geq \mathrm{Brd}_p(F)$*,* $\mathrm{abrd}_p(L) \geq \mathrm{abrd}_p(F)$*,* $\mathrm{Brd}_p(E') = \mathrm{Brd}_p(E)$ *and* $\mathrm{abrd}_p(E') = \mathrm{abrd}_p(E)$*, for each $p \in \mathbb{P}$ different from q.*

Proof. Fix an algebraic closure $\overline{F}$ of F and denote by E_{ins} the perfect closure of E in $\overline{F}$. The extension E_{ins}/E is purely inseparable, so it follows from the Albert-Hochschild theorem (cf. [39], Ch. II, 2.2) that the scalar extension map of $\mathrm{Br}(E)$ into $\mathrm{Br}(E_{\mathrm{ins}})$ is surjective. Since finite extensions of E in E_{ins} are of q-primary degrees, one obtains from (1.1) (c) that $\mathrm{ind}(D \otimes_E E_{\mathrm{ins}}) = \mathrm{ind}(D)$ and $\exp(D \otimes_E E_{\mathrm{ins}}) = \exp(D)$, provided $D \in d(E)$ and $q \dagger \mathrm{ind}(D)$. Therefore, $\mathrm{Brd}_p(E) = \mathrm{Brd}_p(E_{\mathrm{ins}})$ and $\mathrm{abrd}_p(E) = \mathrm{abrd}_p(E_{\mathrm{ins}})$, for each $p \in \mathbb{P}$, $p \neq q$. As $\mathcal{G}_{E_{\mathrm{ins}}} \cong \mathcal{G}_E$ (see [23], Ch. VII, Proposition 12) and $FE_{\mathrm{ins}}/E_{\mathrm{ins}}$ is an FG-extension, this reduces the proof of Proposition 6.1 to the case where E is perfect. It is known (cf. [13], Theorems 12.4.1 and 12.4.2) that then there exists a Henselian field (K, v) with $\mathrm{char}(K) = 0$ and $\widehat{K} \cong E$, which can be chosen so that $v(K) = \mathbb{Z}$ and $v(q) = 1$. Moreover, it follows from (3.4), [28] and Galois theory (see also the proof of [13], Corollary 22.2.3) that there is $E' \in I(K_{\mathrm{sep}}/K)$, such that $E' \cap K_{\mathrm{ur}} = K$ and $E'K_{\mathrm{ur}} = K_{\mathrm{sep}}$. This ensures that $v(E') = \mathbb{Q}$, $\widehat{E}' = \widehat{K} = E$ and $E'_{\mathrm{ur}} = E'_{\mathrm{sep}} = K_{\mathrm{sep}}$. Hence, by (3.3) and (3.5), $\mathcal{G}_{E'} \cong \mathcal{G}_E$, $\mathrm{Brd}_p(E') = \mathrm{Brd}_p(E)$ and $\mathrm{abrd}_p(E') = \mathrm{abrd}_p(E)$, $p \in \mathbb{P} \setminus \{q\}$. Observe that, since E is perfect, F/E is separably generated, i.e. there is $F_0 \in I(F/E)$, such that F_0/E is rational and $F \in \mathrm{Fe}(F_0)$ (cf. [23], Ch. X). Note further that each rational extension L_0 of E' with $\mathrm{trd}(L_0/E') = \mathrm{trd}(F_0/E)$ has a restricted Gauss valuation ω_0 extending $v_{E'}$ with $\widehat{L}_0 = F_0$ (cf. [13], Example 4.3.2). Fixing (L_0, ω_0), one can take its valued extension (L, ω) so that $L_\omega \cong L \otimes_{L_0} L_{0,\omega_0}$ is an inertial lift of F over L_{0,ω_0}. This yields $\omega(L) = \omega_0(L_0) = \mathbb{Q}$, $\widehat{L} \cong F$ over F_0, $[L\colon L_0] = [F\colon F_0]$ and $\mathrm{trd}(L/K) = \mathrm{trd}(F/E)$. It also becomes clear that, for each $F' \in \mathrm{Fe}(F)$, there exists a valued extension (L', ω') of (L, ω) with $[L'\colon L] = [F'\colon F]$ and $\widehat{L}' \cong F'$. Observing now that L'/E', $F' \in \mathrm{Fe}(F)$, are FG-extensions, applying (3.3) and (3.5) to a Henselization $L'_{\omega'}$, for any admissible F', and using Lemmas 3.1 and 4.1, one concludes that $\mathrm{Brd}_p(L') \geq \mathrm{Brd}_p(F')$ and $\mathrm{abrd}_p(L) \geq \mathrm{abrd}_p(F)$, for all $p \in \mathbb{P} \setminus \{q\}$. Proposition 6.1 is proved. $\square$

We show that in zero characteristic Proposition 2.2 can be deduced from Proposition 6.1.

Example 6.2. *Let K_0 be a field with 2 elements, $K_n = K_0((X_1))\dots((X_n))$, $n \in \mathbb{N}$, a sequence of iterated formal power series fields in n variables over K_0, inductively defined by the rule $K_n = K_{n-1}((X_n))$, for each $n \in \mathbb{N}$, and let Θ be a perfect closure of the union $K_\infty = \cup_{n=1}^{\infty} K_n$. It is known that the natural $\mathbb{Z}^n$-valued valuations, say v_n, of the fields K_n, $n \in \mathbb{N}$, extend uniquely to a Henselian K_0-valuation v of K_∞ with $\widehat{K}_\infty = K_0$ and $v(K_\infty) = \cup_{n=1}^{\infty} v_n(K_n)$. Since $r_p(K_0) = 1$, $p \in \mathbb{P}$, and finite extensions of K_∞ in Θ are totally ramified and of 2-primary degrees over K_∞, one deduces from [7], Lemma 4.4, that $Brd_p(K_\infty) = \mathrm{Brd}_p(\Theta) = 1$ and $abrd_p(K_\infty) = \mathrm{abrd}_p(\Theta) = \infty$, for every $p > 2$. At the same time, it follows from Lemma 4.2 that $r_2(\Theta) = \infty$. Hence, by Proposition 6.1, there is a field Θ' with $char(\Theta) = 0$, $abrd_2(\Theta') = 0$, $r_2(\Theta') = \infty$, and $Brd_p(\Theta') = 1$, $abrd_p(\Theta') = \infty$, $p > 2$. Moreover, by the proof of Proposition 6.1, Θ' can be chosen so that its roots of unity form a multiplicative 2-group. Put $\Theta_0 = \Theta'$, $\Theta_k = \Theta_{k-1}((T_k))$, $k \in \mathbb{N}$, and let θ_k be the natural (Henselian) $\mathbb{Z}^k$-valued Θ_0-valuation of Θ_k, for each index k. Fix a maximal extension E_k of Θ_k in $\Theta_{k,\mathrm{sep}}$ with respect to the property that finite extensions of Θ_k in E_k have odd degrees and are totally ramified over Θ_k relative to θ_k. This ensures that $\widehat{E}_k = \Theta_0$, E_k does not contain a primitive μ-th root of unity, for any odd $\mu > 1$, the group $\theta_k(E_k)/2\theta_k(E_k)$ has order 2^k, and $\theta_k(E_k) = p\theta_k(E_k)$, for every $p > 2$. Therefore, by [7], Lemma 4.4, $Brd_2(E_k) = \mathrm{abrd}_2(K) = k$, and by (3.5), $Brd_p(E_k) = 1$ and $abrd_p(E_k) = \infty$, $p > 2$, whence $Brd(E_k) = k$.*

Similarly to Remark 5.5, the proofs of Proposition 6.1 and our concluding result demonstrate the applicability of restricted Gauss valuations in finding lower bounds on $\mathrm{Brd}_p(F)$, for FG-extensions F of valued fields E with $\mathrm{abrd}_p(E) < \infty$:

Proposition 6.3. *Let E be a local field and F/E an* FG*-extension. Then* $\mathrm{Brd}_p(F) \geq 1 + \mathrm{trd}(F/E)$*, for every* $p \in \mathbb{P}$.

Proof. As $\mathrm{Brd}_p(F) = 1$ when $\mathrm{trd}(F/E) = 0$, we assume that $\mathrm{trd}(F/E) = n \geq 1$. We show that, for each $p \in \mathbb{P}$, there exists $D_p \in d(F)$, such that $\exp(D_p) = p$, $\mathrm{ind}(D_p) = p^{n+1}$ and D_p decomposes into a tensor product of cyclic division F-algebras of degree p. Let ω be the standard discrete valuation of E, $\widehat{E}$ its residue field, and F_0 a rational extension of E in F with $\mathrm{trd}(F_0/E) = n$. Considering a discrete restricted Gauss valuation of F_0 extending ω, and its prolongations on F, one obtains that F has a discrete valuation v extending ω, such that $\widehat{F}$ is an FG-extension of $\widehat{E}$ with $\mathrm{trd}(\widehat{F}/\widehat{E}) = n$. Hence, by the proof of Proposition 5.9, given in [6], there exist $\Delta_p' \in d(\widehat{F})$ and a degree p cyclic extension $L_p'/\widehat{F}$, such that $\Delta_p' \otimes_{\widehat{F}} L_p' \in d(L_p')$, $\exp(\Delta_p') = p$, $\mathrm{ind}(\Delta_p') = p^n$ and Δ_p' is a tensor product of cyclic division $\widehat{F}$-algebras of degree p. Given a Henselization $(F_v, \bar{v})$ of (F, v), Lemma 3.1 implies the existence of $\Delta_p \in d(F)$, such that $\Delta_p \otimes_F F_v \in d(F_v)$ is an inertial lift of Δ_p' over F_v. Also, by Lemma 3.2, there is a degree p cyclic extension L_p/F with $L_p \otimes_F F_v$ an inertial lift of L_p' over F_v. Fix a generator σ of $\mathcal{G}(L_p/F)$, take a uniform element β of (F, v), and put $D_p = \Delta_p \otimes_F (L_p/F, \sigma, \beta)$. Then it follows from (3.1) and [31], Theorem 1, that $D_p \in d(F)$, $\exp(D_p) = p$, $\mathrm{ind}(D_p) = p^{n+1}$ and $D_p \otimes_F F_v \in d(F_v)$, so Proposition 6.3 is proved. □

Note finally that if E is a local field, F/E is an FG-extension and $\mathrm{trd}(F/E) = 1$, then $\mathrm{Brd}_p(F) = 2$, for every $p \in \mathbb{P}$. When $p = \mathrm{char}(E)$, this is implied by Proposition 6.3 and Theorem 2.1 (c), and for a proof in the case of $p \neq \mathrm{char}(E)$, we refer the reader to [33], Theorems 1 and 3, [37] and [25], Corollary 1.4.

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